

ABOUT THE CONGRUENCE $\sum_{k=1}^n k^{f(n)} \equiv 0 \pmod{n}$

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ABSTRACT. In this paper we characterize, in terms of the prime divisors of n , the pairs (k, n) for which n divides $\sum_{j=1}^n j^k$. As an application, we study the sets $\mathcal{M}_f := \{n : n \text{ divides } \sum_{j=1}^n j^{f(n)}\}$ for some choices of f .

1. INTRODUCTION

In the literature on power sums

$$S_k(n) := \sum_{j=1}^n j^k$$

the following congruence is well known

Proposition 1. *Let p be a prime and let $k > 0$ be an integer. Then, we have:*

$$S_k(p) \equiv \begin{cases} -1 & \text{if } p-1 \mid k \\ 0 & \text{if } p-1 \nmid k \end{cases} \pmod{p}.$$

Proof. See ([6]) for the standard proof using primitive roots, or ([7]) for a recent elementary proof. \square

The following proposition gives a more general result for $S_k(n)$.

Proposition 2. *(Carlitz-von Staudt, 1961, [2]). Let $k > 1$ and n be positive integers with n even, then*

$$S_k(n) \equiv - \sum_{\substack{p \mid n \\ p-1 \mid k}} \frac{n}{p} \pmod{n}.$$

These results motivate an interest in studying $S_k(n) \pmod{n}$ and, more generally, in studying $S_{f(n)}(n) \pmod{n}$ for different arithmetic functions f . Thus, if $p-1 \mid f(p)$, for every prime p , we have that the congruence $S_{f(n)}(n) \equiv -1 \pmod{n}$ holds for every $n = p$ prime and it is interesting to find the composite numbers which also satisfy it. In this direction we have the *Giuga numbers* (see [1]), which are composite numbers such that $S_{\phi(n)}(n) \equiv -1 \pmod{n}$, the *strong Giuga numbers*, which are composite numbers such that $S_{n-1}(n) \equiv -1 \pmod{n}$ (Giuga's conjecture [3] states that there are no strong Giuga numbers), or the *K-strong Giuga numbers*, which are composite numbers such that $S_{K(n-1)}(n) \equiv -1 \pmod{n}$ (see [4]).

In this paper we characterize, in terms of the prime divisors of n , the pairs (k, n) for which n divides $S_k(n)$. This characterization is given in the following theorem.

Theorem 1. *Let n and k be any integers. Then, $S_k(n) \equiv 0 \pmod{n}$ if and only if one of the following holds:*

- i) n is odd and $p - 1 \nmid k$ for every p prime divisor of n .
- ii) n is a multiple of 4 and $k > 1$ is odd.

Moreover, inspired in Giuga's ideas we investigate the congruence $S_{f(n)}(n) \equiv 0 \pmod{n}$ for some functions f . This work started in [5]) with the case $f(n) = \frac{(n-1)}{2}$. It will be of special interest the case of arithmetic functions f such that $p - 1 \nmid f(p)$ for any prime p .

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. In what follows we will consider the following subset of \mathbb{N} associated to f :

$$\mathcal{M}_f := \{n : n \text{ divides } S_{f(n)}(n)\}$$

We have studied the sets \mathcal{M}_f in the affine case ($f(n) = an + b$) and in some cases such that \mathcal{M}_f contains the set of prime numbers. We have characterized the elements of these sets and, in some cases, we have computed their asymptotic density.

2. SUM OF THE n INITIAL k -TH POWERS MODULO n

Recall that, given two integers n and k , we define $S_k(n) := \sum_{j=1}^n j^k$. In this section we present the main results of the paper. In particular we will characterize the pairs (n, k) such that n divides $S_k(n)$. If $k = 0$, clearly $S_k(n) = n$ and there is no problem to study. Thus, in what follows we will assume $k \neq 0$.

We will start this section with a technical lemma.

Lemma 1. *Let $k > 0$ be any integer and let p be a prime such that $p - 1 \mid k$. Then, for every $m > 0$:*

$$\sum_{j=1}^{p^m} j^k \equiv -p^{m-1} \pmod{p^m}.$$

Proof. Clearly $\sum_{j=1}^{p^m} j^k \equiv \sum_{\substack{1 \leq j \leq p^m \\ (j, p^m)=1}} j^k \pmod{p^m}$, with this latter sum consisting of $p^{m-1}(p-1)$ summands. Since $p-1 \mid k$, if a and b are such that $p \nmid a, b$ then $a^k - b^k \equiv 0 \pmod{p}$. This implies that these summands are the elements of the arithmetic sequence $\{1, p+1, \dots, p(p^{m-1}-1)+1\}$, where every element appears exactly $p-1$ times. Consequently:

$$\begin{aligned} \sum_{j=1}^{p^m} j^k &\equiv \sum_{\substack{1 \leq j \leq p^m \\ (j, p^m)=1}} j^k = (p-1) \sum_{i=1}^{p^{m-1}-1} 1 + ip = (p-1) \frac{p^{m-1}[1 + (1 + p^m - p)]}{2} \\ &\equiv (p-1)p^{m-1} \equiv -p^{m-1} \pmod{p^m}. \end{aligned}$$

□

With the help of this lemma we can prove the following proposition

Proposition 3. *Let n be an odd integer and k be any integer. Then n divides $S_k(n)$ if and only if $\gcd(k, p-1) < p-1$ for every p , prime divisor of n .*

Proof. Put $n = p_1^{r_1} \cdots p_s^{r_s}$ the prime decomposition of n .

Assume that there exists $i \in \{1, \dots, s\}$ such that $p_i - 1 = \gcd(k, p_i - 1)$; i.e., such that $p_i - 1 | k$. Then, since $S_k(n) = \sum_{j=1}^n j^k \equiv \frac{n}{p_i^{r_i}} \sum_{j=1}^{p_i^{r_i}} j^k \pmod{p_i^{r_i}}$, it follows by the previous lemma that $S_k(n) \equiv \frac{n}{p_i^{r_i}} (p_i - 1) p_i^{r_i - 1} \not\equiv 0 \pmod{p_i^{r_i}}$ and n does not divide $S_k(n)$.

Conversely, let us assume that $\gcd(k, p_i - 1) < p_i - 1$ for every i . We claim that $\sum_{\substack{(j,n)=1 \\ 1 \leq j \leq n}} j^k \equiv 0 \pmod{n}$. Let $i \in \{1, \dots, s\}$ and let α be a generator of $\mathcal{U}(\mathbb{Z}/p_i^{r_i}\mathbb{Z})$, we put $x = \alpha^k$. If $x - 1$ was not a unit, then $p_i | x - 1 = \alpha^k - 1$. Since $p_i | \alpha^{p_i - 1} - 1$ it follows that $\alpha^d \equiv 1 \pmod{p_i}$ with $d = \gcd(k, p_i - 1)$. Consequently $\alpha^{dp_i^{r_i - 1}} \equiv 1 \pmod{p_i^{r_i}}$ which is impossible since $dp_i^{r_i - 1} < \varphi(p_i^{r_i})$. Thus:

$$\sum_{\substack{(j,p_i^{r_i})=1 \\ 1 \leq j \leq p_i^{r_i}}} j^k \equiv \frac{x^{\varphi(p_i^{r_i})+1} - x}{x - 1} \equiv 0 \pmod{p_i^{r_i}}.$$

Moreover, since $\sum_{\substack{(j,n)=1 \\ 1 \leq j \leq n}} j^k \equiv \varphi\left(\frac{n}{p_i^{r_i}}\right) \sum_{\substack{(j,p_i^{r_i})=1 \\ 1 \leq j \leq p_i^{r_i}}} j^k \equiv 0 \pmod{p_i^{r_i}}$ and this holds for every i , the claim follows.

Now, let d be any divisor of n . We have that $\sum_{\substack{(j,n)=d \\ 1 \leq j \leq n}} j^k = d^k \sum_{\substack{(j,n/d)=1 \\ 1 \leq j \leq n/d}} j^k$ so it is

enough to apply the previous considerations and to sum over d in order to complete the proof. \square

Now we will turn to the even case. This is done in the following propositions.

Proposition 4. *Let n be an integer with $n \equiv 2 \pmod{4}$ and let k be any integer. Then:*

- i) $S_k(n) \not\equiv 0 \pmod{n}$.
- ii) $S_k(n) \equiv 0 \pmod{\frac{n}{2}}$ if and only if $\gcd(k, p - 1) < p - 1$ for every p , odd prime divisor of n .

Proof. i) We have that $j^k \equiv 0, 1 \pmod{2}$ if j even or odd respectively. This implies that $S_k(n) \equiv \frac{n}{2} \not\equiv 0 \pmod{2}$.
 ii) Proposition 3 implies that $S_k\left(\frac{n}{2}\right) \equiv 0 \pmod{\frac{n}{2}}$ if and only if $\gcd(k, p - 1) < p - 1$ for every p prime divisor of $\frac{n}{2}$. Since $S_k(n) \equiv 2S_k\left(\frac{n}{2}\right) \pmod{\frac{n}{2}}$, the result follows. \square

Proposition 5. *Let n be a multiple of 4 and let k be an odd integer. Then:*

- i) If $k = 1$, $S_k(n) \not\equiv 0 \pmod{n}$.
- ii) If $k > 1$, $S_k(n) \equiv 0 \pmod{n}$.

Proof. The first part is obvious. For the second part, put $n = 2^m n'$ with $m > 1$ and n' being odd.

Since k is odd it follows that $\gcd(k, p-1) < p-1$ for every p prime divisor of n' and Proposition 1 implies that $S_k(n) \equiv 0 \pmod{n'}$. On the other hand, we have that $S_k(n) \equiv 2^m S_k(n') \equiv 0 \pmod{n'}$.

Moreover, $S_k(n) \equiv n' S_k(2^m) \pmod{2^m}$. Now, $k > 1$ being odd, if $j \in \{1, \dots, 2^m - 1\}$ we have that $j^k \equiv -(2^m - j)^k \pmod{2^m}$ so $S_k(2^m) \equiv (2^{m-1})^k \equiv 0 \pmod{2^m}$ and the result follows. \square

Proposition 6. *Let n be a multiple of 4 and let k be an even integer. Then:*

- i) $S_k(n) \not\equiv 0 \pmod{n}$.
- ii) $S_k(n) \equiv 0 \pmod{\frac{n}{2}}$ if and only if $\gcd(k, p-1) < p-1$ for every p , odd prime divisor of n .

Proof. i) Put $n = 2^m n'$ with $m > 1$ and n' being odd.

Again $S_k(n) \equiv n' S_k(2^m) \pmod{2^m}$ and, since k is even, we have that $j^k \equiv (2^m - j)^k \pmod{2^m}$. This implies that $S_k(2^m) \equiv 2 S_k(2^{m-1}) \pmod{2^m}$. This allows us to reason inductively and conclude that:

$$S_k(2^m) \equiv 0 \Leftrightarrow S_k(1) \equiv 0 \pmod{2}.$$

Since the latter is false the result follows.

- ii) By Proposition 1 we have that $S_k(n) \equiv 2^m S_k(n') \equiv 0 \pmod{n'}$ if and only if $\gcd(k, p-1) < p-1$ for every p prime divisor of n' . Now $S_k(n) \equiv n' S_k(2^m) \pmod{2^m}$. Clearly $S_k(1) = 1$ and if $a > 1$ we have that $S_k(2^a) \equiv 2 S_k(2^{a-1}) \pmod{2^a}$. This implies that $S_k(2^m) \equiv 2^{m-1} \pmod{2^m}$ and, consequently, $S_k(n) \equiv n' S_k(2^m) \equiv n' 2^{m-1} = \frac{n}{2} \pmod{2^m}$ and the proof is complete. \square

All the previous work can be summarized in the following theorem.

Theorem 2. *Let n and k be any integers. Then, $S_k(n) \equiv 0 \pmod{n}$ if and only if one of the following holds:*

- i) n is odd and $p-1 \nmid k$ for every p prime divisor of n .
- ii) n is a multiple of 4 and $k > 1$ is odd.

We have just characterized the pairs (n, k) such that n divides $S_k(n)$. It follows immediately from this characterization that, given $n \in \mathbb{N}$ the complement of the set $\mathcal{W}_n := \{k : S_k(n) \equiv 0 \pmod{n}\}$ is:

$$\mathbb{N} \setminus \mathcal{W}_n = \begin{cases} 2\mathbb{N} \cup \{1\} & \text{if } n \equiv 0 \pmod{4}, \\ \mathbb{N} & \text{if } n \equiv 2 \pmod{4}, \\ \bigcup_{p|n} (p-1)\mathbb{N} & \text{if } n \equiv 1, 3 \pmod{4}. \end{cases}$$

In the same way, for every $k \in \mathbb{N}$ the complement of $\mathcal{H}_k := \{n : S_k(n) \equiv 0 \pmod{n}\}$ consists of a finite union of arithmetic sequences. Namely, if we denote $\mathcal{P}_k := \{p \text{ odd prime} : p-1 \mid k\}$, we have that:

$$\mathbb{N} \setminus \mathcal{H}_k = \begin{cases} 2\mathbb{N} & \text{if } k = 1, \\ \bigcup_{p \in \mathcal{P}_k} 2p(2\mathbb{N} + 1) & \text{if } k > 1 \text{ is odd,} \\ \bigcup_{p \in \mathcal{P}_k} p\mathbb{N} & \text{if } k \text{ is even.} \end{cases}$$

Now, we could consider cases when $k = f(n)$ depends on n and then we will be interested in characterizing the values of n such that $S_{f(n)}(n) \equiv 0 \pmod{n}$. This will be done in the following sections for various choices of the function f .

3. THE AFFINE CASE

In this section we will focus in the case when f is a affine function; i.e., $f(n) = an + b$. In what follows we will denote by $f_{a,b}(n) := an + b$. Recall that we defined $\mathcal{M}_f := \{n : n \text{ divides } S_{f(n)}(n)\}$. In what follows it will be easier to characterize the complement $\mathbb{N} \setminus \mathcal{M}_f$ instead of \mathcal{M}_f itself.

Let us introduce some notation. Given $(a, b) \in \mathbb{N} \times \mathbb{Z}$, we will consider the set:

$$\mathcal{P}_{a,b} := \{p \text{ odd prime} : b \equiv 0 \pmod{\gcd(ap, p-1)}\}.$$

and if $(a, b, p) \in \mathbb{N} \times \mathbb{Z} \times \mathcal{P}_{a,b}$ we define

$$\Xi(a, b, p) := a^{-1} \min\{x \in \mathbb{N} : x \equiv 0 \pmod{ap}, x \equiv -b \pmod{p-1}\}.$$

With this notation in mind we can prove the following result [8].

Theorem 3. *Let $(a, b) \in \mathbb{N} \times \mathbb{Z}$. Then:*

i) *If a and b are even,*

$$\mathbb{N} \setminus \mathcal{M}_{f_{a,b}} = 2\mathbb{N} \cup \bigcup_{p \in \mathcal{P}_{a,b}} \{\Xi(a, b, p) + \frac{s}{a} \text{lcm}(ap, p-1) : s \in \mathbb{N}\}.$$

ii) *If a and b are odd,*

$$\mathbb{N} \setminus \mathcal{M}_{f_{a,b}} = \{n : n \equiv 2 \pmod{4}\} \cup \bigcup_{p \in \mathcal{P}_{a,b}} \{\Xi(a, b, p) + \frac{s}{a} \text{lcm}(ap, p-1) : s \in \mathbb{N}\}.$$

iii) *If a is even and b is odd, then*

$$\mathbb{N} \setminus \mathcal{M}_{f_{a,b}} = \{n : n \equiv 2 \pmod{4}\}.$$

iv) *If a is odd and b is even, then*

$$\mathbb{N} \setminus \mathcal{M}_{f_{a,b}} = 2\mathbb{N}.$$

Proof. We will give a complete proof of i), the other cases being analogous.

First observe that a and b being even, then $f_{a,b}(n)$ is even. Consequently, by Theorem 2, we have that $2\mathbb{N} \subseteq \mathbb{N} \setminus \mathcal{M}_{f_{a,b}}$.

Now, assume that $n \notin \mathcal{M}_{f_{a,b}}$ is odd. Then, by Theorem 2 again, there must exist an odd prime $p \mid n$ such that $p-1 \mid an+b$. Since $an \equiv 0 \pmod{ap}$ and $an \equiv -b \pmod{p-1}$ it readily follows that $an \in \{A + (s)\text{lcm}(ap, p-1) : s \in \mathbb{N}\}$ with $A = \min\{x \in \mathbb{N} : x \equiv 0 \pmod{ap}, x \equiv -b \pmod{p-1}\}$. Since it is obvious that A is a multiple of ap we have that $n \in \{\frac{A}{a} + \frac{(s)}{a} \text{lcm}(ap, p-1) : s \in \mathbb{N}\}$ with $\frac{A}{a} = \Xi(a, b, p)$ by definition. To finish the proof it is enough to observe that if $ap \mid an$ and $p-1 \mid an+b$, then $p \in \mathcal{P}_{a,b}$ as claimed. \square

Here and throughout, we denote by $\delta(A)$ (resp. $\underline{\delta}(A)$, $\overline{\delta}(A)$) the asymptotic (resp. upper, lower asymptotic) density of an integer sequence A . We will be interested in computing the asymptotic density of the sets $\mathcal{M}_{f_{a,b}}$, at least for some particular values of a and b . To do so we must first show that this density exists and the following lemma will be our main tool.

Lemma 2. *Let $\mathcal{A} := \{a_k\}_{k \in \mathbb{N}}$ and $\{c_k\}_{k \in \mathbb{N}}$ be two sequences of positive integers and $\mathcal{B}_k := \{a_k + (s-1)c_k : s \in \mathbb{N}\}$. If $\sum_{k=1}^{\infty} \frac{1}{c_k}$ is convergent and \mathcal{A} has zero asymptotic*

density, then $\bigcup_{k=1}^{\infty} \mathcal{B}_k$ has an asymptotic density with:

$$\delta\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) = \lim_{n \rightarrow \infty} \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right)$$

and

$$\delta\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) - \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right) \leq \sum_{i=n+1}^{\infty} \frac{1}{c_i}.$$

Proof. Let us denote $B_n := \bigcup_{k=n+1}^{\infty} \mathcal{B}_k$ and $\vartheta(n, N) := \text{card}([0, N] \cap B_n)$. Then:

$$\vartheta(n, N) \leq \text{card}([0, N] \cap \mathcal{A}) + N \sum_{k=n+1}^{\infty} \frac{1}{c_k}.$$

From this, we get:

$$\bar{\delta}(B_n) = \limsup \frac{\vartheta(n, N)}{N} \leq \limsup \frac{\text{card}([0, N] \cap \mathcal{A})}{N} + \sum_{k=n+1}^{\infty} \frac{1}{c_k} = \sum_{k=n+1}^{\infty} \frac{1}{c_k}.$$

Now, for every n , $\bigcup_{k=1}^n \mathcal{B}_k$ has an asymptotic density and the sequence $\delta_n := \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right)$ is non-decreasing and bounded (by 1), thus convergent. Consequently:

$$\begin{aligned} \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right) &\leq \underline{\delta}\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) \leq \bar{\delta}\left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) = \bar{\delta}\left(\bigcup_{k=1}^n \mathcal{B}_k \cup B_n\right) \\ &\leq \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right) + \bar{\delta}(B_n) \leq \delta\left(\bigcup_{k=1}^n \mathcal{B}_k\right) + \sum_{k=n+1}^{\infty} \frac{1}{c_k}, \end{aligned}$$

and considering that $\sum_{k=n+1}^{\infty} \frac{1}{c_j}$ converges to zero it is enough to take limits in order to finish the proof. \square

With the help of this lemma the following proposition is easy to prove.

Proposition 7. *For every $a, b \in \mathbb{N}$, $\mathcal{M}_{f_{a,b}}$ has an asymptotic density and it is a computable number.*

Proof. It is enough to see that $\mathbb{N} \setminus \mathcal{M}_{f_{a,b}}$ has an asymptotic density and that it is a computable number.

Cases ii) y iii) above are obvious. In cases i) and iv) it is enough to apply the previous lemma since $\mathbb{N} \setminus \mathcal{M}_{f_{a,b}}$ is a countable union of arithmetic sequences whose initial terms $(\Xi(a, b, p))$ form a set of zero asymptotic density, and the series $\sum_{p \text{ prime}} \frac{a}{\text{lcm}(ap, p-1)}$ is convergent. \square

The rest of this section will be devoted to study $\delta(\mathcal{M}_{f_{a,b}})$ in some particular cases. Namely, the cases $(a, b) = (1, b)$. When b is even, $\mathcal{M}_{f_{1,b}}$ is exactly the set of odd integers and its asymptotic density is $\frac{1}{2}$. The case when b is odd is much more

interesting. In particular we will see that, in this case, the asymptotic density of $\mathcal{M}_{f_{1,b}}$ is slightly greater than $\frac{1}{2}$.

In the following lemma we give a more explicit description of the elements of $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$ with odd b . This description will be useful to compute $\delta(\mathcal{M}_{f_{1,b}})$.

Lemma 3. *Let n be an integer and let $b \in \mathbb{Z}$ be odd. Then $n \in \mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$ if and only if $n \equiv 2 \pmod{4}$ or n is odd and it is of the form $kp^2 - kp - bp$ for some p odd prime and $\frac{b}{p-1} < k \in \mathbb{Z}$. In other words, if $\mathcal{G}_p^b := \mathbb{N} \cap \{-bp \pmod{p(p-1)}\}$, we have that:*

$$\mathbb{N} \setminus \mathcal{M}_{f_{1,b}} = \bigcup_{p \geq 3} \mathcal{G}_p^b \cup \{2 \pmod{4}\}.$$

Proof. Let $n \in \mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$. Then, by Theorem 2, $n \equiv 2 \pmod{4}$ or it is odd and there exists p prime divisor of n such that $p-1$ divides $n+b$. Put $n = pm$, then $n+b = pm+b = (p-1)m + m-b$ so $p-1$ must divide $m-b$ and $m = k(p-1) + b$ for some k and $n = pm = kp^2 - kp - bp$ as claimed.

The converse is obvious due to Theorem 2 again. \square

In order to compute $\delta(\mathcal{M}_{f_{1,b}}) = 1 - \delta(\mathbb{N} \setminus \mathcal{M}_{f_{1,b}})$ with the help of lemmata 2 and 3 and of the Principle of Inclusion and Exclusion it will be necessary to have a good criterion to determine when the intersection of \mathcal{G}_p^b for various odd primes p is empty. Put

$$\mathcal{R}_b := \{m > 2 : \gcd(m, \phi(m)) \text{ divides } b\}.$$

Thus, we have the following result.

Proposition 8. *Let \mathcal{P} be a finite set of primes and put $m := \prod_{p \in \mathcal{P}} p$. Then $\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b$ is nonempty if and only if $m \in \mathcal{R}_b$, where this set is defined above. If this is the case, then the set $\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b$ is an arithmetic progression of difference $\text{lcm}(m, \lambda(m))$.*

Proof. It is clear that $\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b$ is nonempty if and only if there exists n such that $n \equiv -b \pmod{p-1}$ and $n \equiv 0 \pmod{p}$ for every $p \in \mathcal{P}$. This happens if and only if there exists n such that $n \equiv -b \pmod{\lambda(m)}$ and $n \equiv 0 \pmod{m}$ and this set of congruences have a solution if and only if $\gcd(m, \lambda(m))$ divides b . To finish the proof it is enough to observe that, m being square-free, $\gcd(m, \lambda(m)) = \gcd(m, \phi(m))$ and apply the Chinese Remainder Lemma. \square

To compute the density of the set $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$ we consider $3 = p_1 < p_2 < \dots$ the increasing sequence of all the odd primes and $k := k(\varepsilon)$ minimal such that

$$\sum_{j \geq k} \frac{1}{p_j(p_j-1)} < \varepsilon.$$

Thus, with an error of at most ε , the density of the set $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$ is the same as the density of $\bigcup_{j < k} \mathcal{G}_{p_j}^b$:

$$\delta \left(\bigcup_{j < k} \mathcal{G}_{p_j}^b \right) < \delta(\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}) < \delta \left(\bigcup_{j < k} \mathcal{G}_{p_j}^b \right) + \varepsilon$$

and, by the Principle of Inclusion and Exclusion,

$$\delta \left(\bigcup_{j < k} \mathcal{G}_{p_j}^b \right) = \sum_{s \geq 1} \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq k-1} \frac{\varepsilon_{i_1, i_2, \dots, i_s}}{\text{lcm}[p_{i_1}(p_{i_1}-1), \dots, p_{i_s}(p_{i_s}-1)]},$$

with the coefficient $\varepsilon_{i_1, i_2, \dots, i_s}$ being zero if $\bigcap_{t=1}^s \mathcal{G}_{p_{i_t}}^b = \emptyset$, and being $(-1)^{s-1}$ otherwise. In other terms, putting $\Pi_k := \prod_{i=2}^k p_i$,

$$\delta \left(\bigcup_{j < k} \mathcal{G}_{p_j}^b \right) = - \sum_{\substack{m | \Pi_k \\ m \in \mathcal{R}_b}} \frac{(-1)^{\omega(m)}}{\text{lcm}(m, \lambda(m))}$$

where, $\omega(m)$ is the number of distinct prime factors of m .

In the case $b = \pm 1$ the asymptotic density of $\mathcal{M}_{f_{1,b}}$ is closely related to that of the set $\mathfrak{P} := \{n \in \mathbb{N} \text{ odd} : S_{\frac{n-1}{2}} \equiv 0 \pmod{n}\}$ which was defined and studied in [5]. In this previous work $\delta(\mathfrak{P})$ was computed up to 3 digits: 0.379... More specifically, it was seen that $\delta(\mathfrak{P}) \in [0.379005, 0.379826]$.

Proposition 9. *For $b \in \{-1, 1\}$ the following holds:*

$$\delta(\mathcal{M}_{f_{1,b}}) = 2\delta(\mathfrak{P}) - \frac{1}{4} \in [0.50801, 0.50966].$$

Proof. Let \mathbb{I} denote the set of odd positive integers. For any odd prime p let us define the following set:

$$\mathcal{F}_p := \{p^2 \pmod{2p(p-1)}\},$$

and recall the definition:

$$\mathcal{G}_p^b := \mathbb{N} \cap \{-bp \pmod{p(p-1)}\}.$$

In [5] it was seen that:

$$\mathbb{I} \setminus \mathfrak{P} = \bigcup_{p \geq 3} \mathcal{F}_p$$

and in the previous proposition we have just proved that:

$$\mathbb{N} \setminus \mathcal{M}_{f_{1,b}} = \bigcup_{p \geq 3} \mathcal{G}_p^b \cup \{2 \pmod{4}\}.$$

If $b = \pm 1$ and for every prime p we have $\delta(\mathcal{G}_p^b) = 2\delta(\mathcal{F}_p)$. Reasoning in a way similar to that in [5], we can see that for every odd b and every set of primes \mathcal{P} it holds:

$$2\delta \left(\bigcap_{p \in \mathcal{P}} \mathcal{F}_p \right) = \delta \left(\bigcap_{p \in \mathcal{P}} \mathcal{G}_p^b \right).$$

Consequently:

$$\delta(\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}) = \frac{1}{4} + \delta \left(\bigcup_{p \geq 3} \mathcal{G}_p^b \right) = \frac{1}{4} + 2\delta(\mathbb{I} \setminus \mathfrak{P}) = \frac{1}{4} + 2 \left(\frac{1}{2} - \delta(\mathfrak{P}) \right)$$

and finally, since $\delta(\mathfrak{P})$ belongs to $[0.379005, 0.379826]$ we obtain that:

$$\delta(\mathcal{M}_{f_{1,b}}) = 1 - \delta(\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}) = 2\delta(\mathfrak{P}) - 1/4 \in [0.50801, 0.50966].$$

□

It is easy to observe that if b is odd and $|b| > 1$ then $\delta(\mathcal{M}_{f_{1,b}}) > \delta(\mathcal{M}_{f_{1,1}})$. Moreover, if b and b' are odd with $|b| \neq |b'|$ and b divides b' then $\delta(\mathcal{M}_{f_{1,b'}}) > \delta(\mathcal{M}_{f_{1,b}})$. In addition it is also easy to observe that the supremum of the densities $\delta(\mathcal{M}_{f_{1,b}})$ is:

$$\mathfrak{S} := \lim_{k \rightarrow \infty} \sum_{m \mid \Pi_k} \frac{(-1)^{\omega(m)}}{\text{lcm}(m\lambda(m))} - \frac{1}{4}.$$

since this is a decreasing sequence any value of k will provide an upper bound for \mathfrak{S} . Computing the value for $k = 22$, we can say that for every odd $b \neq \pm 1$

$$0.50801 < \delta(\mathcal{M}_{f_{1,1}}) < \delta(\mathcal{M}_{f_{1,b}}) < \mathfrak{S} < 0.647.$$

We will say that a positive integer n is an *anti-Korselt number* if for every p prime divisor of n , $p - 1$ does not divide $(n - 1)$. This section will be closed computing the asymptotic density of anti-Korselt numbers. In order to do this we observe that Theorem 2 gives the following characterization.

Lemma 4. *An integer n is an anti-Korselt number if and only if $\sum_{j=1}^n j^{n-1} \equiv 0 \pmod{n}$ and $4 \nmid n$.*

Proof. Just apply Theorem 2 and observe that, by definition, anti-Korselt numbers are odd. \square

Proposition 10. *The set of anti-Korselt numbers has asymptotic density whose value is:*

$$2\delta(\mathfrak{P}) - \frac{1}{2} \in [0.25801, 0.259652].$$

Proof. By the previous lemma the set of anti-Korselt numbers is $\mathfrak{K} := \mathcal{M}_{f_{1,-1}} \setminus 4\mathbb{N}$. Since, $4\mathbb{N} \subset \mathcal{M}_{f_{1,-1}}$, it follows that

$$\delta(\mathfrak{K}) = \delta(\mathcal{M}_{f_{1,-1}}) - \frac{1}{4} = 2\delta(\mathfrak{P}) - \frac{1}{2}$$

as claimed. \square

4. \mathcal{M}_f CONTAINING THE PRIME NUMBERS

In this section we will characterize the set \mathcal{M}_f for some functions f such that $f(p) = \frac{p-1}{2}$ for every odd prime. Note that in this case \mathcal{M}_f contains all odd primes. In particular, we will focus on $f = \frac{\varphi}{2}$ and $f = \frac{\lambda}{2}$, where φ and λ denote Euler and Carmichael function, respectively.

Proposition 11. $\mathcal{M}_{\frac{\varphi}{2}} = \{p^k : p \text{ odd prime}\}.$

Proof. If p is an odd prime and $k \in \mathbb{N}$, $\frac{\varphi(p^k)}{2} = \frac{p^{k-1}(p-1)}{2}$ and $\gcd\left(\frac{p^{k-1}(p-1)}{2}, p-1\right) < p-1$. Consequently we can apply Proposition 1 to get that $p^k \in \mathcal{M}_{\frac{\varphi}{2}}$.

Now, if n is odd and there exists p, q distinct odd primes dividing n it readily follows that $p-1$ divides $\frac{\varphi(n)}{2}$ so Proposition 3 applies and it follows that $n \notin \mathcal{M}_{\frac{\varphi}{2}}$. Thus, if an odd $n \in \mathcal{M}_{\frac{\varphi}{2}}$ it must be $n = p^k$.

Finally, if $n \in \mathcal{M}_{\frac{\varepsilon}{2}}$ is even Proposition 4 implies that 4 divides n and Proposition 6 implies that $\frac{\varphi(n)}{2}$ is odd. Since these statements are contradictory the result follows. \square

In what follows we will use the notation $\mathcal{E}(m) := \max\{k \in \mathbb{N} : 2^k \text{ divides } m\}$.

Proposition 12. *Let $n = 2^m p_1^{r_1} \cdots p_s^{r_s}$ with $s > 0$. Then $n \in \mathcal{M}_{\frac{\lambda}{2}}$ if and only if one of these conditions holds:*

- i) $m = 0$ and $\mathcal{E}(p_i - 1) = \mathcal{E}(p_j - 1)$ for every i, j .
- ii) $m = 2$ or 3 , $\mathcal{E}(p_i - 1) = 1$ for every i and $\frac{n}{2^m} \neq 3$.

Proof. If condition i) holds, $n = p_1^{r_1} \cdots p_s^{r_s}$ and $p_i = 2^t q_i + 1$ with q_i even and t not depending on i . In this case $\lambda(n) = \text{lcm}(\varphi(p_1^{r_1}), \dots, \varphi(p_s^{r_s})) = 2^t \text{lcm}(p_1^{r_1-1} q_1, \dots, p_s^{r_s-1} q_s) = 2^t L$ with L odd. Consequently $\frac{\lambda(n)}{2} = 2^{t-1} L$ and since L is odd it follows that $p_i - 1$ does not divide $\frac{\lambda(n)}{2}$ and Proposition 3 implies that $n \in \mathcal{M}_{\frac{\lambda}{2}}$.

If condition ii) holds, it follows that $\lambda(n) = 2L$ with $L > 1$ odd. Consequently $\frac{\lambda(n)}{2} = L > 1$ is odd and Proposition 5 applies to conclude that $n \in \mathcal{M}_{\frac{\lambda}{2}}$.

Finally, assume that $n = 2^m p_1^{r_1} \cdots p_s^{r_s}$ with $s > 0$ and $p_i = 2^{m_i} q_i + 1$ with q_i odd is such that $n \in \mathcal{M}_{\frac{\lambda}{2}}$. First of all, Proposition 4 implies that $m = 0$ or $m > 1$.

If $m > 1$, Proposition 5 (i) implies that $\frac{n}{2^m} \neq 3$ and Proposition 6 implies that $\frac{\lambda(n)}{2}$ is odd so $m = 2$ or 3 and $p_i^{r_i-1}(p_i - 1) = \varphi(p_i^{r_i}) = 2L_i$ with L_i odd; i.e., $p_i - 1 = 2q_i$ with q_i odd as claimed.

If, on the other hand, $m = 0$, Proposition 3 implies that $p_i - 1$ does not divide $\frac{\lambda(n)}{2}$ for any i . But if $m_i > m_j$ for some $i \neq j$ we have that $2^{m_i-1} q_j$ divides $\frac{\lambda(n)}{2}$ and, consequently, $p_j - 1$ divides $\frac{\lambda(n)}{2}$. A contradiction. \square

Before we proceed we will introduce some notation and technical results. Given a prime p and a subset $A \subseteq \mathbb{N}$, we define the set:

$$A_p := \{n \in A : p \mid n \text{ but } p^2 \nmid n\}.$$

With this notation we have the following result.

Lemma 5. *If for a set of primes $\{p_i\}_{i \in I}$ we have $\delta(A_{p_i}) = 0$ for every $i \in I$, and $\sum_{i \in I} p_i^{-1} = \infty$, then $\delta(A) = 0$.*

Now, given a positive integer k we define the set:

$$\Upsilon_k := \{n \text{ odd} : \mathcal{E}(p - 1) = k \text{ for every } p \mid n\}.$$

With this notation, Proposition 11 states that:

$$\mathcal{M}_{\frac{\lambda}{2}} = \left(\bigcup_{k=1}^{\infty} \Upsilon_k \cup 4\Upsilon_1 \cup 8\Upsilon_1 \right) \setminus \{12, 24\}.$$

We are in the condition to compute the asymptotic density of $\mathcal{M}_{\frac{\lambda}{2}}$.

Proposition 13. *$\mathcal{M}_{\frac{\lambda}{2}}$ has zero asymptotic density.*

Proof. Since

$$\mathcal{M}_{\frac{\lambda}{2}} = \left(\bigcup_{k=1}^{\infty} \Upsilon_k \cup 4\Upsilon_1 \cup 8\Upsilon_1 \right) \setminus \{12, 24\},$$

it will be enough to show that $A = \bigcup_{n=1}^{\infty} \Upsilon_n$ has zero asymptotic density.

For any prime p let us introduce the following sets:

$$\begin{aligned}\mathcal{I}_p &:= \{p\} \cup \{q \text{ prime} : \mathcal{E}(p-1) \neq \mathcal{E}(q-1)\}, \\ \mathcal{T}_p &:= \{n \in \mathbb{N} : p \nmid n\}, \\ \mathcal{K}_p &:= \{pk : k \in \mathbb{N}\}.\end{aligned}$$

It is easy to observe that, for any prime p :

$$\mathbb{N} \setminus A_p = \bigcup_{q \in \mathcal{I}_p} p\mathcal{K}_q \cup \mathcal{T}_p.$$

Now, considering that

$$\mathcal{I}_p = \{p\} \bigcup \{q \text{ prime} : q \not\equiv 2^k + 1 \pmod{2^{k+1}} \text{ with } k = \mathcal{E}(p-1)\}$$

it is clear that $\delta\left(\bigcup_{q \in \mathcal{I}_p} \mathcal{K}_q\right) = 1$. Thus, $\delta\left(\bigcup_{q \in \mathcal{I}_p} p\mathcal{K}_q\right) = \frac{1}{p}$ and since $\delta(\mathcal{T}_p) = \frac{p-1}{p}$ it follows that, for any prime p :

$$\delta(A_p) = 1 - \delta(\mathbb{N} \setminus A_p) = 1 - \delta\left(\bigcup_{q \in \mathcal{I}_p} p\mathcal{K}_q\right) - \delta(\mathcal{T}_p) = 0$$

and the result follows from the previous lemma. \square

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